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Long-time behavior of a model of extensible beams with nonlinear boundary dissipations

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ABSTRACT

This paper is concerned with the long-time behavior of a model of extensible beams

$$u_{tt} + u_{xxxx} - M\left(\int_0^L |u_x|^2\,dx\right) u_{xx} = h,$$

defined in a bounded interval $(0, L)$, where M is a continuous nonnegative function and h is a static load. To this problem little is known with nonlinear boundary conditions. One considers the case where a nonlinear forcing interacts with the shear force at the boundary. Then the existence of a global attractor is proved with a sole boundary damping.

1. Introduction

This work is concerned with the long-time behavior of solutions, $u = u(x, t)$, of a fourth order equation

$$u_{tt} + u_{xxxx} - M(\|u_x(t)\|_2^2)u_{xx} = h(x), \quad (x, t) \in (0, L) \times \mathbb{R}^+, \tag{1.1}$$

with nonlinear boundary conditions

$$u(0, t) = u_x(0, t) = 0, \quad t > 0, \tag{1.2}$$

$$u_{xx}(L, t) = 0, \qquad u_{xxx}(L, t) - M(\|u_x(t)\|_2^2)u_x(L, t) = f(u(L, t)) + g(u_t(L, t)), \quad t > 0, \tag{1.3}$$

and initial conditions

$$u(x, 0) = u^0(x), \qquad u_t(x, 0) = u^1(x), \quad x \in (0, L), \tag{1.4}$$

where

$$\|u_x(t)\|_2^2 = \int_0^L |u_x(x, t)|^2\,dx.$$

Eq. (1.1) is a model for vibrations of extensible beams where u stands for the vertical deflections. The extensibility effects are represented by the term $M(\|u_x(t)\|_2^2)u_{xx}$. This was proposed by Woinowsky-Krieger [1] in the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4} - \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.5}$$

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where L, E, I, ρ, H and A denote, respectively, the length of the beam in the rest position, Young's modulus, the cross-sectional moment of inertia, the mass density, the tension in the rest position, and the cross-sectional area. Other works on the modeling aspects of extensible beams and plates can be found in [2–4]. In particular, in Giorgi et al. [5,6] one can find mathematical modeling for viscoelastic and thermoelastic versions of extensible beams.

In our case, condition (1.2) means that the beam is clamped at its left end-point. On the other hand, the first condition in (1.3) means that the bending moment at the end-point $x = L$ is null. The second condition in (1.3) is related to the shear forces at $x = L$ and shows its dependence on $f(u(L, t))$ and $g(u_t(L, t))$. Then, as a physical interpretation, the right hand side of the beam is supported by a spring, represented by f, and subject to an external dissipation represented by g. Indeed, the energy of the system is defined by

$$\mathcal{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u_{xx}(t)\|_2^2 + \frac{1}{2}\int_0^{\|u_x(t)\|_2^2} M(z)dz + \int_0^{u(L,t)} f(z)dz,$$

where we have assumed $h = 0$ for simplicity. Multiplying (1.1) by u_t and integrating over $(0, L)$ it follows that

$$\frac{d}{dt}\mathcal{E}(t) = -g(u_t(L, t))u_t(L, t).$$

This shows that the energy of the system is decreasing if we assume $g(z)z \geq 0, \forall z \in \mathbb{R}$.

From the view point of mathematical analysis, extensible beams with null boundary conditions were considered by several authors. We quote, for instance, Ball [8,9], Dickey [10,11] and Medeiros [12], considered as pioneering works. Other interesting contributions can be found in, for instance, [13,14,5,6,15–19] and the references therein.

The second order (in time) evolution equations with nonlinear boundary conditions have been considered by several well-known authors; e.g. [20–26], among others. Now, with respect to extensible beams with nonlinear boundary conditions involving the third order spatial-derivative, there are only a few works in the literature. One of the first studies in this direction was done by Pazoto and Perla Menzala [27], where stabilization of a thermoelastic extensible beam was considered. Motivated by that result, Ma [7] studied the boundary stabilization of (1.1)–(1.4) with $h = 0$. We also refer the reader to some closely related works. In [28,29], boundary feedback control for problem (1.1)–(1.4) was studied. In [30,31], the Euler–Bernoulli equations were considered with nonlocal boundary dissipations acting on third derivatives. In addition, an extensible beam with dynamic boundary conditions was considered by Grobbelaar-Van Dalsen and van der Merwe [15].

The objective of the present work is to provide a result on the existence of global attractors to the problem (1.1)–(1.4) under essentially the same hypotheses in [7]. Therefore we extend or complement the results in [30,15,28,29,7,31,27].

This paper is organized as follows. In Section 2 we introduce the assumptions and discuss the existence of global strong and weak solutions of the problem (1.1)–(1.4). In Section 3, we establish our main result, Theorem 3.2, which guarantees the existence of a global attractor for (1.1)–(1.4) in $H^2(0, L) \times L^2(0, L)$.

2. Assumptions and the global existence

In this section we present our assumptions on the functions M, f, g and discuss the well-posedness of the problem (1.1)–(1.4).

We assume that $M \in C^1(\mathbb{R})$ satisfying

$$M(z)z \geq \widehat{M}(z) \geq 0, \quad \forall z \geq 0, \tag{2.6}$$

where $\widehat{M}(z) = \int_0^z M(s)ds$. This condition is promptly satisfied if M is nondecreasing with $M(0) \geq 0$. The functions $f, g : \mathbb{R} \to \mathbb{R}$ are also of class $C^1(\mathbb{R})$ and satisfy

$$f(0) = g(0) = 0. \tag{2.7}$$

There exist constants $k, k_1 k_2, L_0, L_1 > 0$ and $\sigma, r \geq 0$ such that

$$|f(u) - f(v)| \leq k(1 + |u|^\sigma + |v|^\sigma)|u - v|, \quad \forall u, v \in \mathbb{R}, \tag{2.8}$$

$$-L_0 \leq \widehat{f}(u) \leq \frac{1}{2}f(u)u + L_1, \quad \forall u \in \mathbb{R}, \tag{2.9}$$

where $\widehat{f}(z) = \int_0^z f(s)ds$,

$$(g(u) - g(v))(u - v) \geq k_1|u - v|^2, \quad \forall u, v \in \mathbb{R}, \tag{2.10}$$

and

$$|g(u) - g(v)| \leq k_2(1 + |u|^r + |v|^r)|u - v|, \quad \forall u, v \in \mathbb{R}. \tag{2.11}$$

With respect to the Sobolev spaces which will be used, let

$$V = \{u \in H^2(0, L);\ u(0) = u_x(0) = 0\}$$

and

$$W = \{u \in H^4(0, L) \cap V;\ u_{xx}(L) = 0\}.$$

Motivated by the boundary condition (1.3) we assume, for regular solutions, that data (u^0, u^1) satisfy the compatibility condition

$$u^0_{xxx}(L) - M(\|u^0_x\|_2^2)u^0_x(L) = f(u^0(L)) + g(u^1(L)). \tag{2.12}$$

Then for regular solutions we consider the phase space

$$\mathcal{H}_1 = \{(u^0, u^1) \in W \times W;\ \text{satisfies condition (2.12)}\}. \tag{2.13}$$

In the case of weak solutions we consider the phase space

$$\mathcal{H}_0 = \overline{\mathcal{H}_1}^{V\times L^2}, \tag{2.14}$$

which guarantees that for regular data, the compatibility condition (2.12) holds. In $\mathcal{H}_0$ we adopt the norm defined by

$$\|(u, v)\|^2_{\mathcal{H}_0} = \|(u, v)\|^2_{V\times L^2} = \|u_{xx}\|_2^2 + \|v\|_2^2.$$

Theorem 2.1. *Assume that conditions (2.6)–(2.11) hold and that $h \in L^2(0, L)$. Then for any initial data*

$$(u^0, u^1) \in \mathcal{H}_1,$$

there exists a unique regular solution u of the problem (1.1)–(1.4) such that

$$u \in L^\infty_{loc}(\mathbb{R}^+; W) \cap C^0([0, \infty); V) \cap C^1([0, \infty); L^2(0, L)).$$

In addition,

$$\|u_t(t)\|_2^2 + \|u_{xx}(t)\|_2^2 + \widehat{M}(\|u_x(t)\|_2^2) \le M_1, \tag{2.15}$$

where $M_1 > 0$ depends on the initial data and h, but not on $t > 0$. If the initial data

$$(u^0, u^1) \in \mathcal{H}_0,$$

there exists a unique weak solution of problem (1.1)–(1.4) which depends continuously on initial data with respect to the norm of $V \times L^2(0, L)$.

Remark 1. When $h = 0$, the global existence and exponential stability of (finite energy) regular solutions of problem (1.1)–(1.4) was studied by Ma [7] under essentially the same hypotheses as above. The existence of weak solutions follows from standard density arguments as shown in [30]. Then Theorem 2.1 follows from [7,30] with obvious changes in order to handle the new independent term $h = h(x)$.

Remark 2. Theorem 2.1 implies that problem (1.1)–(1.4) defines a nonlinear C_0-semigroup $S(t)$ on $\mathcal{H}_0$. Indeed, let us set $S(t)(u^0, u^1) = (u(t), u_t(t))$ where u is the unique weak solution corresponding to initial data (u^0, u^1). Because of the well-posedness of the problem, the semigroup property will be satisfied if we show that $S(t)$ maps $\mathcal{H}_0$ into $\mathcal{H}_0$. To this end, given $(u^0, u^1) \in \mathcal{H}_0$, there exists a sequence $(u^0_\mu, u^1_\mu) \in \mathcal{H}_1$ such that

$$u^0_\mu \to u^0 \quad \text{in } V \quad \text{and} \quad u^1_\mu \to u^1 \quad \text{in } L^2(0, L).$$

Then there exists a sequence of regular solutions (u_μ) such that $(u_\mu(t), u_{\mu t}(t))$ converges strongly to $(u(t), u_t(t))$ in $V \times L^2(0, L)$, for any $t > 0$. Now, due to the regularity, the boundary condition (1.3) makes sense and implies that the compatibility condition (2.12) holds for $(u_\mu(t), u_{\mu t}(t))$, for any $t > 0$. Therefore, for any $t > 0$, $S(t)(u^0, u^1)$ is the $V \times L^2$-limit of a sequence $(u_\mu(t), u_{\mu t}(t)) \in \mathcal{H}_1$, that is, $S(t)(u^0, u^1) \in \mathcal{H}_0$.

3. Global attractor

A global attractor for a C_0-semigroup $S(t)$ defined on a complete metric space H, is a bounded closed subset $\mathcal{A} \subset H$ which is positive fully invariant, that is, $S(t)\mathcal{A} = \mathcal{A}, \forall\, t \ge 0$, and uniformly attracting, that is,

$$\operatorname{dist}(S(t)B, \mathcal{A}) = \sup_{x\in S(t)B} \inf_{y\in\mathcal{A}} d(x, y) \to 0 \quad \text{as } t \to \infty,$$

for any bounded set $B \subset H$.

A bounded set $\mathcal{B} \subset H$ is an absorbing set for $S(t)$ if for any bounded set $B \subset H$, there exists $t_B = t(B) \ge 0$ such that

$$S(t)B \subset \mathcal{B}, \quad \forall\, t \ge t_B,$$

which defines $(H, S(t))$ as a dissipative dynamical system. A semigroup $S(t)$ is asymptotically smooth in H if for any bounded positive invariant set $B \subset H$, there exists a compact set $K \subset \overline{B}$, such that

$$dist(S(t)B, K) \to 0 \quad \text{as } t \to \infty.$$

Then the following theorem is well-known (see e.g. [32,21,33,34]).

Theorem 3.1 *([21], Theorem 2.3)*. *Let $S(t)$ be a dissipative C_0-semigroup defined on a metric space H. Then $S(t)$ has a compact global attractor in H if and only if it is asymptotically smooth in H.*

The asymptotic smoothness can be verified from a result by Khanmamedov [35] and Chueshov and Lasiecka [21]. Assume that H is a Banach space.

Lemma 3.1 *([21], Proposition 2.10)*. *Assume that for any bounded positive invariant set $B \subset H$, and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$\|S(T)x - S(T)y\| \le \varepsilon + \phi_T(x, y), \quad \forall\, x, y \in B, \tag{3.16}$$

where $\phi_T : H \times H \to \mathbb{R}$ satisfies

$$\liminf_{k\to\infty} \liminf_{l\to\infty} \phi_T(z_k, z_l) = 0, \tag{3.17}$$

for any sequence (z_n) of B. Then $S(t)$ is asymptotically smooth in H.

Our main result reads as follows.

Theorem 3.2. *Assume the hypotheses of Theorem 2.1 with $\sigma = r = 0$ and $k > 0$ sufficiently small. Then the corresponding semigroup $S(t)$ of problem (1.1)–(1.4) has a compact global attractor in the phase space $\mathcal{H}_0$.*

Remark 3. (a) Since $\sigma = 0$ we have $|f(u)| \le 3k|u|$ for all $u \in \mathbb{R}$. Then from our arguments a sufficient smallness condition for k is

$$1 - 9kL^3 > 0, \tag{3.18}$$

where the dependence on L is due to embedding constants (3.20) below. (b) In order to control the nonlinear terms on the boundary we strategically use the embedding of $H^1(0, L)$ into $L^\infty(0, L)$. Therefore our result cannot be directly extended to bigger dimensions.

Proof of Theorem 3.2. Following Theorem 3.1, we show that semigroup $S(t)$ has an absorbing set and that satisfies the asymptotic smoothness property in $\mathcal{H}_0$. To this end, by density, we may formally assume that solutions of problem (1.1)–(1.4) are regular, so that we can perform the necessary calculations. In particular, we can calculate the derivative of the perturbed total energy functional

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|u_{xx}(t)\|_2^2 + \widehat{M}(\|u_x(t)\|_2^2)\right) + \widehat{f}(u(L, t)) - \int_0^L hu(t)dx. \tag{3.19}$$

Also, since $u(0) = u_x(0) = u_{xx}(L) = 0$, the following inequalities hold:

$$\|u\|_\infty \le \sqrt{L}\|u_x\|_2, \quad \|u_x\|_\infty \le \sqrt{L}\|u_{xx}\|_2, \quad \|u_x\|_2 \le L\|u_{xx}\|_2, \quad \|u\|_2 \le L^2\|u_{xx}\|_2. \tag{3.20}$$

Step 1. The existence of an absorbing set in $\mathcal{H}_0$.

Let us fix an arbitrary bounded set $B \subset \mathcal{H}_0$ and consider the solutions of problem (1.1)–(1.4) given by $(u(t), u_t(t)) = S(t)(u_0, u_1)$ with $(u_0, u_1) \in B$. Our analysis is based on the modified energy functional

$$\widetilde{E}(t) = E(t) + L^4\|h\|_2^2 + L_0. \tag{3.21}$$

It is easy to see that $\widetilde{E}(t)$ dominates $\|(u(t), u_t(t))\|^2_{V\times L^2}$. Indeed, since $\|u(t)\|_2 \le L^2\|u_{xx}(t)\|_2$ one has

$$\int_0^L hu(t)dx \le L^2\|h\|_2\|u_{xx}(t)\|_2 \le L^4\|h\|_2^2 + \frac{1}{4}\|u_{xx}(t)\|_2^2.$$

Since $\widehat{f}(u(L, t)) \ge -L_0$, we get

$$\widetilde{E}(t) \ge \frac{1}{4}\left(\|u_{xx}(t)\|_2^2 + \|u_t(t)\|_2^2\right) = \frac{1}{4}\|(u(t), u_t(t))\|^2_{V\times L^2}. \tag{3.22}$$

Let us define

$$\psi(t)=\int_0^L u_t(t)xu_x(t)dx.$$

We show that for $\varepsilon>0$ small enough,

$$\frac{d}{dt}\widetilde{E}(t)+\varepsilon\frac{d}{dt}\psi(t)+\varepsilon\widetilde{E}(t)\le\varepsilon C_h,\quad \forall\, t\ge 0, \tag{3.23}$$

where $C_h>0$ is a constant depending on h, but not on $\{u^0,u^1\}$.

By multiplying Eq. (1.1) by u_t and integrating over $(0,L)$, we have

$$\frac{d}{dt}E(t)+g(u_t(L,t))u_t(L,t)=0,\quad t\ge 0. \tag{3.24}$$

Then, multiplying Eq. (1.1) by (xu_x) and integrating over $(0,L)$, we obtain

$$\begin{aligned}&\frac{d}{dt}\psi(t)+\frac{1}{2}\|u_t(t)\|_2^2+\frac{3}{2}\|u_{xx}(t)\|_2^2+\frac{1}{2}M(\|u_x(t)\|_2^2)\|u_x(t)\|_2^2\\&\quad=\frac{L}{2}|u_t(L,t)|^2+\int_0^L hxu_x(t)dx-Lu_{xxx}(L,t)u_x(L,t)+\frac{L}{2}M(\|u_x(t)\|_2^2)|u_x(L,t)|^2.\end{aligned} \tag{3.25}$$

Taking the sum of (3.24) with ε times (3.25), we have

$$\begin{aligned}&\frac{d}{dt}E(t)+\varepsilon\frac{d}{dt}\psi(t)+\varepsilon\left(\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|u_{xx}(t)\|_2^2+\frac{1}{2}M(\|u_x(t)\|_2^2)\|u_x(t)\|_2^2\right)\\&\qquad+\varepsilon\|u_{xx}(t)\|_2^2+g(u_t(L,t))u_t(L,t)\\&\quad=\frac{\varepsilon L}{2}|u_t(L,t)|^2+\varepsilon\int_0^L hxu_x(t)dx-\varepsilon Lu_{xxx}(L,t)u_x(L,t)+\frac{\varepsilon L}{2}M(\|u_x(t)\|_2^2)|u_x(L,t)|^2.\end{aligned} \tag{3.26}$$

Adding $\varepsilon\left(\widehat{f}(u(L,t))-\int_0^L hu(t)dx+L_0+L^4\|h\|_2^2\right)$ on both sides of the equality (3.26) and taking into account that $\frac{d}{dt}E(t)=\frac{d}{dt}\widetilde{E}(t)$ and assumption (2.10), we obtain

$$\begin{aligned}&\frac{d}{dt}\widetilde{E}(t)+\varepsilon\frac{d}{dt}\psi(t)+\varepsilon\widetilde{E}(t)+\varepsilon\|u_{xx}(t)\|_2^2+k_1|u_t(L,t)|^2\\&\quad\le\varepsilon L_0+\varepsilon L^4\|h\|_2^2+\frac{\varepsilon L}{2}|u_t(L,t)|^2+\varepsilon\widehat{f}(u(L,t))-\varepsilon\int_0^L hudx\\&\qquad+\varepsilon\int_0^L hxu_x(t)dx-\varepsilon Lg(u_t(L,t))u_x(L,t)-\varepsilon Lf(u(L,t))u_x(L,t).\end{aligned} \tag{3.27}$$

In what follows we estimate the terms on the right hand side of (3.27).

(i) By assumption (2.9) and (3.20) we have

$$\begin{aligned}\varepsilon\widehat{f}(u(L,t))&\le\frac{\varepsilon}{2}f(u(L,t))u(L,t)+\varepsilon L_1\\&\le\frac{\varepsilon}{2}3k\|u(t)\|_\infty^2+\varepsilon L_1\\&\le\frac{\varepsilon}{2}3kL^3\|u_{xx}(t)\|_2^2+\varepsilon L_1.\end{aligned} \tag{3.28}$$

(ii) Using (3.20) we have

$$\begin{aligned}-\varepsilon\int_0^L hu(t)dx+\varepsilon\int_0^L hxu_x(t)dx&\le 2\varepsilon L^2\|h\|_2\|u_{xx}(t)\|_2\\&\le\varepsilon(4L^4)\|h\|_2^2+\frac{\varepsilon}{4}\|u_{xx}(t)\|_2^2.\end{aligned} \tag{3.29}$$

(iii) Using (2.11) and (3.20), we obtain

$$\begin{aligned}-\varepsilon Lg(u_t(L,t))u_x(L,t)&\le\varepsilon 3k_2L^{\frac{3}{2}}|u_t(L,t)|\|u_{xx}(t)\|_2\\&\le\varepsilon 9k_2^2L^3|u_t(L,t)|^2+\frac{\varepsilon}{4}\|u_{xx}(t)\|_2^2.\end{aligned} \tag{3.30}$$

(iv) Finally, using assumption (2.8) and (3.20) we have

$$
\begin{aligned}
-\varepsilon L f(u(L,t))u_x(L,t) &\le \varepsilon 3kL\|u(t)\|_\infty\|u_x(t)\|_\infty \\
&\le \varepsilon 3kL^3\|u_{xx}(t)\|_2^2.
\end{aligned}
\tag{3.31}
$$

Then inserting (3.28)–(3.31) into (3.27) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\widetilde{E}(t)+\varepsilon\frac{d}{dt}\psi(t)+\varepsilon\widetilde{E}(t)+\frac{\varepsilon}{2}\left(1-9kL^3\right)\|u_{xx}(t)\|_2^2 \\
&\quad +k_1\left(1-\varepsilon\left(\frac{L}{2}+9k_2^2L^3\right)\right)|u_t(L,t)|^2 \le \varepsilon(L_0+L_1+5L^4\|h\|_2^2).
\end{aligned}
$$

In view of (3.18), taking $\varepsilon>0$ small enough we get (3.23).

Now, let us define

$$
\widetilde{E}_\varepsilon(t)=\widetilde{E}(t)+\varepsilon\psi(t).
$$

Then from (3.22),

$$
\begin{aligned}
\left|\widetilde{E}_\varepsilon(t)-\widetilde{E}(t)\right| &\le \varepsilon L^2\|u_t(t)\|_2\|u_{xx}(t)\|_2 \\
&\le \varepsilon 2L^2\widetilde{E}(t),
\end{aligned}
$$

which implies that, for ε small enough,

$$
\frac{1}{2}\widetilde{E}(t)\le\widetilde{E}_\varepsilon(t)\le\frac{3}{2}\widetilde{E}(t).
\tag{3.32}
$$

Inserting (3.32) into (3.23) we obtain

$$
\frac{d}{dt}\widetilde{E}_\varepsilon(t)+\frac{2\varepsilon}{3}\widetilde{E}_\varepsilon(t)\le\varepsilon C_h.
$$

Applying Gronwall's lemma we get

$$
\widetilde{E}_\varepsilon(t)\le\widetilde{E}_\varepsilon(0)e^{-\frac{2\varepsilon}{3}t}+\frac{3}{2}C_h\left(1-e^{-\frac{2\varepsilon}{3}t}\right),
$$

and using (3.32) again, we conclude that

$$
\widetilde{E}(t)\le 3\widetilde{E}(0)e^{-\frac{\varepsilon}{3}t}+3C_h\left(1-e^{-\frac{\varepsilon}{3}t}\right).
\tag{3.33}
$$

Here we point out that, by density with respect to the norm of $V\times L^2(0,L)$, the energy inequality (3.33) holds for weak solutions. Let us finish the proof for the existence of absorbing sets. Since the given invariant set B is bounded, the initial energy $\widetilde{E}(0)$ is also bounded. Then there exists $t_B>0$, large enough, such that

$$
\widetilde{E}(t)\le 3C_h,\quad \forall\, t>t_B.
$$

Taking into account (3.22) we see that

$$
\|(u(t),u_t(t))\|_{\mathcal{H}_0}^2\le 12C_h,\quad \forall\, t>t_B.
$$

This shows that

$$
\mathcal{B}=\{(u,v)\in\mathcal{H}_0\,;\ \|(u,v)\|_{\mathcal{H}_0}^2\le 12C_h\}
$$

is an absorbing set for $S(t)$ in $\mathcal{H}_0$.

Step 2. Asymptotic Compactness.

We are going to apply Lemma 3.1 somewhat similar to Nakao [36]. Given initial data (u_0,u_1) and (v_0,v_1) in a bounded invariant set $B\subset\mathcal{H}_0$, let u, v be the corresponding weak solutions of the problem (1.1)–(1.4). Then the difference $w=u-v$ is a weak solution of

$$
\begin{cases}
w_{tt}+w_{xxxx}-M(\|u_x(t)\|_2^2)w_{xx}-\Delta_M v_{xx}=0,\\
w(0,t)=w_x(0,t)=w_{xx}(L,t)=0,\\
w_{xxx}(L,t)-M(\|u_x(t)\|_2^2)w_x(L,t)=\Delta_M v_x(L,t)+\Delta_f+\Delta_g,\\
w(x,0)=u_0-v_0,\qquad w_t(x,0)=u_1-v_1,
\end{cases}
\tag{3.34}
$$

where

$$
\begin{aligned}
&\Delta_M:=M(\|u_x(t)\|_2^2)-M(\|v_x(t)\|_2^2),\\
&\Delta_f:=f(u(L,t))-f(v(L,t))\quad\text{and}\quad \Delta_g:=g(u_t(L,t))-g(v_t(L,t)).
\end{aligned}
$$

Let us consider the energy functional

$$F(t) = \frac{1}{2}\|w_t(t)\|_2^2 + \frac{1}{2}\|w_{xx}(t)\|_2^2 + \frac{1}{2}M(\|u_x(t)\|_2^2)\|w_x(t)\|_2^2$$

and define

$$\phi(t) = \int_0^L w_t x w_x dx.$$

Then we show that there exists $\eta > 0$ sufficiently small such that

$$\frac{d}{dt}F(t) + \eta\frac{d}{dt}\phi(t) + \frac{\eta}{2}F(t) \le C|u_t(L,t)|\|w_x(t)\|_2 + C\|w_x(t)\|_2^2, \quad t \ge 0, \tag{3.35}$$

where $C > 0$ is a constant depending on the size of the invariant set B but not on t.

As before, by density, we can assume formally that w is sufficiently regular. Then, multiplying first equation in (3.34) by w_t and integrating over $(0, L)$ we get

$$\frac{d}{dt}F(t) + w_{xxx}(L,t)w_t(L,t) - M\left(\|u_x(t)\|_2^2\right)w_x(L,t)w_t(L,t)$$
$$= M'(\|u_x(t)\|_2^2)\int_0^L u_x u_{xt} dx\|w_x(t)\|_2^2 + \Delta_M\int_0^L v_{xx}w_t dx.$$

Taking into account the third equation in (3.34) we see that

$$\begin{aligned}\frac{d}{dt}F(t) + \Delta_g w_t(L,t) = {} & M'(\|u_x(t)\|_2^2)\int_0^L u_x u_{xt} dx\|w_x(t)\|_2^2 + \Delta_M\int_0^L v_{xx}w_t dx \\ & - \Delta_M v_x(L,t)w_t(L,t) - \Delta_f w_t(L,t).\end{aligned} \tag{3.36}$$

By multiplying first equation in (3.34) by xw_x and integrating over $(0, L)$ we obtain

$$\begin{aligned}&\frac{d}{dt}\phi(t) + \frac{1}{2}\|w_t\|_2^2 + \frac{3}{2}\|w_{xx}\|_2^2 + \frac{1}{2}M(\|u_x(t)\|_2^2)\|w_x(t)\|_2^2 \\ &= \frac{L}{2}|w_t(L,t)|^2 + \Delta_M\int_0^L v_{xx}xw_x dx - Lw_{xxx}(L,t)w_x(L,t) + \frac{L}{2}M(\|u_x(t)\|_2^2)|w_x(L,t)|^2.\end{aligned} \tag{3.37}$$

Then summing (3.36) with η times (3.37) we obtain in view of (2.10) that,

$$\begin{aligned}&\frac{d}{dt}F(t) + \eta\frac{d}{dt}\phi(t) + \eta F(t) + \eta\|w_{xx}(t)\|_2^2 + k_1|w_t(L,t)|^2 \\ &\le \frac{\eta L}{2}|w_t(L,t)|^2 + M'(\|u_x(t)\|_2^2)\int_0^L u_x u_{xt} dx\|w_x(t)\|_2^2 + \Delta_M\int_0^L v_{xx}w_t dx \\ &\quad - \Delta_M v_x(L,t)w_t(L,t) + \eta\Delta_M\int_0^L v_{xx}xw_x dx - \Delta_f w_t(L,t) \\ &\quad - \eta L\Delta_M v_x(L,t)w_x(L,t) - \eta L\Delta_f w_x(L,t) - \eta L\Delta_g w_x(L,t).\end{aligned} \tag{3.38}$$

Let us estimate the right hand side of (3.38). We recall that u, v, w satisfy the estimate (2.15). Then denoting by C_0 a generic positive constant which depends only on B we can simplify several notations. For instance,

$$M'(\|u_x(t)\|_2^2) \le C_0, \quad t \ge 0,$$

since $M \in C^1(\mathbb{R})$ and $\|u_x(t)\|_2^2 \le C_0$. Also, we have by (2.10),

$$\int_0^\infty |u_t(L,t)|^2 dt \le C_0. \tag{3.39}$$

Then we have

$$\begin{aligned}&M'(\|u_x(t)\|_2^2)\int_0^L u_x u_{xt} dx\|w_x(t)\|_2^2 \\ &= M'(\|u_x(t)\|_2^2)\left(u_x(L,t)u_t(L,t) - \int_0^L u_{xx}u_t dx\right)\|w_x(t)\|_2^2 \\ &\le C_0\left(\|u_{xx}(t)\|_2|u_t(L,t)| + \|u_{xx}(t)\|_2\|u_t(t)\|_2\right)\|w_x(t)\|_2^2 \\ &\le C_0|u_t(L,t)|\|w_x(t)\|_2 + C_0\|w_x(t)\|_2^2.\end{aligned} \tag{3.40}$$

Note that from the mean value theorem,

$$\Delta_M \le M_1(\|u_x(t)\|_2 + \|v_x(t)\|_2)\,|\|u_x(t)\|_2 - \|v_x(t)\|_2| \le C_0\|w_x(t)\|_2.$$

Then,

$$\begin{aligned}\Delta_M \int_0^L v_{xx} w_t dx &\le C_0\|w_x(t)\|_2\|w_t(t)\|_2\\ &\le C_0\|w_x(t)\|_2^2 + \frac{\eta}{2}F(t).\end{aligned} \tag{3.41}$$

In addition,

$$\eta\Delta_M \int_0^L v_{xx} x w_x dx \le C_0\|w_x(t)\|_2^2, \tag{3.42}$$

and

$$\begin{aligned}-\Delta_M v_x(L,t)w_t(L,t) &\le C_0\|w_x(t)\|_2|w_t(L,t)|\\ &\le C_0\|w_x(t)\|_2^2 + \frac{k_1}{4}|w_t(L,t)|^2.\end{aligned} \tag{3.43}$$

$$-\eta L\Delta_M v_x(L,t)w_x(L,t) \le C\|w_x(t)\|_2^2. \tag{3.44}$$

From assumption (2.8) on f we get

$$\begin{aligned}\Delta_f w_t(L,t) &\le 3k\|w(t)\|_\infty|w_t(L,t)|\\ &\le C_0\|w_x(t)\|_2^2 + \frac{k_1}{4}|w_t(L,t)|^2,\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}-\eta L\Delta_f w_x(L,t) &\le \eta L3k\|w(t)\|_2\|w_x(t)\|_\infty\\ &\le C_0\|w_x(t)\|_2^2 + \eta^2\|w_{xx}(t)\|_2^2.\end{aligned} \tag{3.46}$$

Finally, from assumption (2.11) on g we have

$$\begin{aligned}-\eta L\Delta_g w_x(L,t) &\le \eta L^2 3k_2|w_t(L,t)|\,\|w_{xx}(t)\|_2\\ &\le \eta^2\frac{L^4 9k_2^2}{k_1}\|w_{xx}(t)\|_2^2 + \frac{k_1}{4}|u_t(L,t)|^2.\end{aligned} \tag{3.47}$$

Substituting (3.40)–(3.47) into (3.38) we obtain

$$\begin{aligned}&\frac{d}{dt}F(t) + \eta\frac{d}{dt}\phi(t) + \frac{\eta}{2}F(t) + \eta\left(1-\eta\left(1+\frac{L^4 9k_2^2}{k_1}\right)\right)\|w_{xx}(t)\|_2^2\\ &\quad\le C_0|u_t(L,t)|\|w_x(t)\|_2 + C_0\|w_x(t)\|_2^2,\end{aligned}$$

which shows the existence of a $\eta > 0$ sufficiently small such that (3.35) holds.

Defining $F_\eta(t) = F(t) + \eta\phi(t)$, by analogous arguments used in the proof of the absorbing set, it follows that for η small,

$$\frac{1}{2}F(t) \le F_\eta(t) \le \frac{3}{2}F(t), \quad t \ge 0,$$

and

$$\frac{d}{dt}F_\eta(t) + \frac{\eta}{3}F_\eta(t) \le C_0|u_t(L,t)|\|w_x(t)\|_2 + C_0\|w_x(t)\|_2^2, \quad t \ge 0.$$

From Gronwall's lemma,

$$\begin{aligned}F_\eta(t) &\le F_\eta(0)e^{-\frac{\eta}{3}t} + C_0\int_0^t e^{-\frac{\eta}{3}(t-s)}|u_t(L,s)|\,\|w_x(s)\|_2\,ds + C_0\int_0^t e^{-\frac{\eta}{3}(t-s)}\|w_x(s)\|_2^2\,ds,\\ &\le F_\eta(0)e^{-\frac{\eta}{3}t} + C_0\left(\int_0^\infty |u_t(L,s)|^2 ds\right)^{1/2}\left(\int_0^t \|w_x(s)\|_2^2\,ds\right)^{1/2} + C_0\int_0^t \|w_x(s)\|_2^2\,ds.\end{aligned}$$

Therefore in view of (3.39) and definition of $F_\eta(t)$, we can fix a constant $C_B > 0$, depending on the size of B but not on $t > 0$, such that

$$\|w(t)\|_{V\times L^2} \le C_B e^{-\frac{\eta}{6}t} + C_B G\left(\int_0^t \|w_x(s)\|_2^2\,ds\right), \tag{3.48}$$

where $G(z) = (z^{1/2} + z)^{1/2}$.

Given $\varepsilon > 0$, we choose T large such that

$$C_B e^{-\frac{\eta}{6}T} < \varepsilon \tag{3.49}$$

and define $\phi_T : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{R}$ as

$$\phi_T((u_0, u_1), (v_0, v_1)) = C_B G\left(\int_0^T \|u_x(t) - v_x(t)\|_2^2\,dt\right).$$

Then from (3.48) and (3.49) we get

$$\|S(T)(u_0, u_1) - S(T)(v_0, v_1)\|_{\mathcal{H}_0} \le \varepsilon + \phi_T((u_0, u_1), (v_0, v_1)), \tag{3.50}$$

for all $(u_0, u_1), (v_0, v_1) \in B$. That is, (3.16) holds.

Let us verify that (3.17) also holds. In fact, let (u_0^n, u_1^n) be a given sequence of initial data in B. Then the corresponding sequence $(u^n(t), u_t^n(t))$ of solutions of the problem (1.1)–(1.4) is uniformly bounded in $\mathcal{H}_0$, because B is bounded and positively invariant. So

$$\{u^n\} \text{ is bounded in } C([0,\infty), V) \cap C^1([0,\infty), L^2(\Omega)).$$

Since $V \hookrightarrow H_0^1(\Omega)$ compactly, there exists a subsequence $\{u^{n_k}\}$ which converges strongly in $C([0,T], H_0^1(\Omega))$. Therefore

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T \|u_x^{n_k}(s) - u_x^{n_l}(s)\|_2^2\,ds = 0,$$

and consequently

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi_T((u_0^{n_k}, u_1^{n_k}), (u_0^{n_l}, u_1^{n_l})) = 0.$$

This shows that (3.17) holds and so the asymptotic smoothness of $S(t)$ follows from Lemma 3.1.

The existence of a global attractor for the semigroup $S(t)$ on $\mathcal{H}_0$ is then a consequence of Theorem 3.1. This ends the proof of Theorem 3.2.

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